Scaled Variables

Often when we're working with process models, we use scaled variables or dimensionless variables. This helps simplify a complex model and helps with further analysis.

- For instance, we can define a scaled heat input:

$$Q_{\text{scaled}} = \frac{Q_s}{F_s C_p} \Rightarrow T_s = T_{is} + Q_{\text{scaled}}$$

- What is the process gain?

$$\frac{2T_s}{3Q_{\text{scaled}}} = 1$$

For every unit change of $Q_{\text{scaled}}$, the tank temperature increases by 1°C (at steady-state).

- Up to this point, we have developed models and looked at steady-state behavior. We will now start looking at process dynamics.

- We can integrate the differential equation:

$$\frac{dT}{dt} = \frac{F_i}{V} (T_i - T) + \frac{Q}{V C_p}$$

For a given change in the heat input ($Q$), we can plot the output temperature change, as a function of the residence time ($V/F$).

Q: What do you expect the dynamics to look like?

- If ($V/F$) is large, will $T$ reach a new steady-state quickly or slowly?

- By looking at the input-output graphs, what is the gain for this process?
Numerical Integration of Process Models

- We have stepped through several examples where we developed process models using balance equations. Now, how do we solve these models?
  - (a) **Analytic integration** → exact solution
  - (b) **Numerical integration**: \( \frac{dT}{dt} = \frac{dx}{dt} = x' = f(x, u, p) \) → state-variable notation

- We can re-write this derivative as: \( \frac{\Delta x}{\Delta t} = \frac{x(k+1) - x(k)}{t(k+1) - t(k)} \) "\( k \)" = time index

\[ \Rightarrow \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{dx}{dt} \]

- We can choose \( \Delta t \) to be very small and solve for the value of \( x \) at some point in the future:

\[ x(k+1) = x(k) + \Delta t \cdot \left( \frac{dx}{dt} \right) = x(k) + \Delta t \cdot f(x, u, p) \]

- Using this form, the equation can be integrated by incrementally stepping from time \( t = \Phi \) \((k=\Phi)\) to the desired end point. We just need an **initial condition**, \( [x @ t = \Phi] \). \( x(\Phi) \to x(1) \to x(2) \to \ldots x(\infty) \).

- This is the way that most computer software works, such as **Matlab**.

### Deviation Variables

- The solutions to a lot of the control problems that we will be dealing with are much easier to handle if you work in terms of deviation variables. Also, this often makes the process much easier to conceptualize.

**Q:** What is a deviation variable?

**A:** A variable minus its steady-state value.

- As an example, let's look again at the stirred tank heater...
Stirred Tank Heater

- Solution from the energy balance: $\frac{dT}{dt} = \frac{F_i}{V_S} (T_{is} - T) + \frac{Q}{V_S \cdot C_P}$

(assuming volume & flowrates are fixed)

→ define deviation variables: $y = T - T_s$  $u = Q - Q_s$

$y$ and $Q_s$ are constants

→ Now, solve for a new expression with these new variables.

\[
\frac{d(y + T_s)}{dt} = \frac{F_i}{V_S} (T_{is} - y - T_s) + \frac{(u + Q_s)}{V_S \cdot C_P}
\]

\[
\frac{dy}{dt} = \frac{F_i}{V_S} (T_{is} - y - (\frac{T_{is} + Q_s}{F_i \cdot C_P})) + \frac{(u + Q_s)}{V_S \cdot C_P}
\]

\[
\frac{du}{dt} = -y (\frac{F_i}{V_S}) - \frac{Q_s}{V_S \cdot C_P} + \frac{u}{V_S \cdot C_P} + \frac{Q_s}{V_S \cdot C_P}
\]

\[
(y \frac{V_S}{F_i}) \frac{dy}{dt} = -y + \left(\frac{1}{F_i \cdot C_P}\right) u
\]

@ s.s. $\Rightarrow$ $y = \left[\frac{u}{k_p}\right] u$

$k_p$ (process gain)

\[
\tau_p \frac{du}{dt} = -y + k_p u
\] → @ s.s. $\Rightarrow \frac{du}{dt} = k_p u$ → $\frac{dy}{du} = k_p$

* One of the most common forms used to model chemical engineering processes.

To solve this model, we need to know: $y$ at steady-state
$u$ at steady-state

Answer: $y = 0$ @ s.s.
$u = 0$ @ s.s. → Nice and Easy!
Isothermal CSTR

- Solution from mass balance, mole balance:

\[
\frac{dC_A}{dt} = \frac{F_i}{V} (C_{A_i} - C_A) - kC_A
\]

→ Define deviation variables:

\[
\begin{align*}
\Delta y &= C_A - C_{A_s} \\
\Delta u &= C_{A_i} - C_{A_s}
\end{align*}
\]

Cas, Cais are constants

→ What is CAs?

\[
\phi = \frac{F_i}{V} (C_{Ais} - C_{A_s}) - kC_{Ais}
\]

\[
C_{Ais} \left(\frac{F_i}{V} + k\right) = \frac{F_i}{V} Cais
\]

\[
C_{Ais} = \frac{\frac{F_i}{V} Cais}{\left(\frac{F_i}{V} + k\right)} = \frac{Cais}{\left(1 + k\frac{V}{F_i}\right)}
\]

→ Substitute variables:

\[
\frac{d(y + C_{Ais})}{dt} = \frac{F_i}{V} (C_{Ais} - (y + C_{Ais}) - k(y + C_{Ais})
\]

\[
\frac{dy}{dt} = \frac{F_i}{V} \left[\frac{u + C_{Ais} - y - \frac{F_i}{V} Cais}{(F_i/V + k)}\right] - ky - k \cdot \frac{F_i}{V} \cdot Cais
\]

\[
\frac{V}{F_i} \frac{dy}{dt} = u + C_{Ais} - y - \frac{(F_i/V) Cais}{(F_i/V + k)} - ky \cdot \frac{V}{F_i} - k Cais \cdot \frac{V}{F_i/V + k}
\]

\[
\frac{V}{F_i} \frac{dy}{dt} = u + C_{Ais} - y - Cais \cdot \left[\frac{(F_i/V) + k}{(F_i/V) + k}\right] - ky \cdot \frac{V}{F_i}
\]

\[
\frac{V}{F_i} \frac{dy}{dt} = u + C_{Ais} - y - Cais \cdot \frac{ky \cdot V}{F_i}
\]

\[
\frac{V}{F_i} \frac{dy}{dt} = u - y \left[1 + \frac{kV}{F_i}\right]
\]

\[
\frac{V}{F_i} \cdot \left[\frac{1}{1 + \frac{kV}{F_i}}\right] \frac{dy}{dt} = -y + \left(\frac{1}{1 + \frac{kV}{F_i}}\right)u
\]

\[
\frac{1}{1 + \frac{kV}{F_i}} \left[\frac{dy}{dt}\right] = -y + \left(\frac{1}{1 + \frac{kV}{F_i}}\right)u
\]
Isothermal CSTR

Now, it is very easy to find relationship between input and output:

At steady-state: \[ \Phi = -y + \left( \frac{1}{1 + k^2} \right) u \]
\[ y = \left( \frac{1}{1 + k^2} \right) u \]

Q1: If \( C_{ai} = C_{ais} \), \( C_a = ? \)

Q2: If \( C_{ai} > C_{ais} \), \( C_a = ? \)

A1: \( C_a = C_{as} \)
A2: \( C_a = C_{as} + \left( \frac{1}{1 + k^2} \right) \Delta u \)

Q: Are the units of the gain correct?