• When you develop models from mass balance, energy balance, mole balance, the models may be non-linear differential equations. Unfortunately, many control strategies are based on linear systems theory.

Q: How do we transform our non-linear models into a usable form?
A: Taylor series approximation

→ Using a Taylor series approximation, we can linearize an equation near its steady-state operating point. Therefore, our approximate solution will be most accurate when we are close to steady-state and less accurate when we have large deviations from steady-state.

• Example: Taylor series approximation with 2 state variable:

\[
\frac{dx}{dt} = \dot{x} = f(x) \quad \{\text{"x" would be } T, h, V, C_x, \text{ etc.}\}
\]

\[
f(x) \approx f(x_s) + \frac{df}{dx} \bigg|_{x_s} (x - x_s) + \frac{1}{2} \frac{d^2f(x)}{dx^2} \bigg|_{x_s} (x - x_s)^2 + \ldots
\]

• We typically neglect quadratic and higher order terms:

\[
f(x_s) = ?
\]

\[
f'(x_s) = \frac{dx}{dt} \bigg|_{x_s} = \Phi
\]

\[
\frac{d(x' + x_s)}{dt} \rightarrow \frac{dx'}{dt}
\]

\[
\frac{dx}{dt} \approx \frac{dF(x)}{dx} \bigg|_{x_s} (x - x_s) \rightarrow \text{define } x' = x - x_s
\]

\[
\frac{dx'}{dt} \approx a x'
\]

\{ \frac{dF(x)}{dx} \bigg|_{x_s} \text{ is a constant} \}
What if we have more than 1 variable?

$\frac{d\gamma}{dt} = f(\gamma, u)$

Taylor series approximation:

$f(x, u) \approx f(x_s, u_s) + \frac{df(x, u)}{dx}\bigg|_{x_s, u_s} (x-x_s) + \frac{df(x, u)}{du}\bigg|_{x_s, u_s} (u-u_s)$

$f(x_s, u_s) = \gamma_s$

$f(x_s, u_s) = \frac{d\gamma}{dt}\bigg|_{x_s, u_s} = 0$

Define deviation variables:

$\gamma' = x-x_s$

$u' = u-u_s$

$\frac{d\gamma'}{dt} = \frac{df}{dx}\bigg|_{x_s, u_s} (\gamma') + \frac{df}{du}\bigg|_{x_s, u_s} (u')$ (a)

(b)

We can do the same thing with the output from our process:

$y = g(x, u) \approx g(x_s, u_s) + \frac{dg}{dx}\bigg|_{x_s, u_s} (x-x_s) + \frac{dg}{du}\bigg|_{x_s, u_s} (u-u_s)$

$(y-y_s) = y_s$ (c)

(y - y_s) = y_s$ (c)

Define deviation variables:

$y' = y-y_s$

$\gamma' = x-x_s$

$u' = u-u_s$

$y' = c\gamma' + du'$
• We can generalize our linearization procedure:

\( \dot{x}' = Ax' + Bu' \) represents our group of differential equations

\( y' = Cx' + Du' \) represents our group of outputs

*Now, \( A, B, C, D \) are groups of constants in matrix form. Once we solve for these constants, we can solve our system of equations fairly easily.

• These constants are defined as:

\[
A_{ij} = \frac{\partial f_i}{\partial x_j} |_{x,s, u,s} \\
B_{ij} = \frac{\partial f_i}{\partial u_j} |_{x,s, u,s} \\
C_{ij} = \frac{\partial g_i}{\partial x_j} |_{x,s, u,s} \\
D_{ij} = \frac{\partial g_i}{\partial u_j} |_{x,s, u,s}
\]

• NOTE: When working with state variable notation, the prime symbol (') is often dropped since it is assumed that the model is always going to be in deviation variable form.

• To see how to apply this method to a real problem, we can look at the gas surge tank example...

\[
\frac{dP}{dt} = \frac{RT}{V} q_i - \frac{RT}{V} \beta VP - P_{atm}
\]

*Assumed \( T = \) constant
• The square root term makes our model non-linear and difficult to solve, so we can linearize our model with a Taylor series approximation.

• For the linearization, we need to know the states and the inputs:
  \[ \dot{x} = ? \]
  \[ \dot{u} = ? \]

  **Answer:** \( \dot{x} = (P) \), \( \dot{u} = (q_i) \)

\[ f(x, u) = \frac{dp}{dt} = \frac{df}{dp} \bigg|_{P_s, q_{is}} (P - P_s) + \frac{df}{q_i} \bigg|_{P_s, q_{is}} (q_i - q_{is}) \]

\[ \frac{dp}{dt} = \frac{-RT}{V} \beta \frac{1}{2(P_s - P_{atm})^2} (P - P_s) + \frac{RT}{V} (q_i - q_{is}) \]

• Define deviation variables:
  \[ P' = P - P_s \]
  \[ q_i' = q_i - q_{is} \]

\[ \frac{dp'}{dt} = \frac{-RT}{V} \beta \frac{1}{2(P_s - P_{atm})^2} (P') + \frac{RT}{V} (q_i') \]

(a) \[ \frac{dp'}{dt} = \alpha P' + b q_i' \]

(b) \[ \dot{x}' = \alpha \dot{x}' + bu' \]
Example 2.4 - CSTR with 2nd Order Reaction

- \( V = \text{constant} \)
- \( C = \text{constant} \)
- \( -r_A = kC_A^2 \)

What is \( \frac{dC_A}{dt} \)?

\[
f = \frac{dC_A}{dt} = \frac{F}{V} (C_{Ai} - C_A) - kC_A^2
\]

What is \( \frac{\partial f}{\partial C_A} \)?

\[
\frac{\partial f}{\partial C_A} = \frac{F}{V} (C_{Ai} - C_A) - 2kC_A
\]

Now linearize our model and transform it into state space notation:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

\[
A = \frac{\partial f}{\partial x} \bigg|_{x_0, u_0}
B = \frac{\partial f}{\partial u} \bigg|_{x_0, u_0}
C = \frac{\partial y}{\partial x} \bigg|_{x_0, u_0}
D = \frac{\partial y}{\partial u} \bigg|_{x_0, u_0}
\]

\[
\begin{align*}
a_{11} &= \frac{\partial f}{\partial x_1} \bigg|_{x_0, u_0} = -\frac{F}{V} - 2kC_A^2 \\
b_{11} &= \frac{\partial f}{\partial u_1} \bigg|_{x_0, u_0} = \frac{F}{V} \\
b_{12} &= \frac{\partial f}{\partial u_2} \bigg|_{x_0, u_0} = \frac{C_{Ai} - C_A}{V}
\end{align*}
\]

\[
\begin{align*}
c_{11} &= 1 \\
d_{11} &= \phi = \frac{\partial g_1}{\partial u_1} = \frac{d(C_A)}{dC_{Ai}} = 0 \\
d_{12} &= \phi = \frac{d(C_A)}{dF} = 0 \\
x' &= C_A - C_A \\
u_1 &= C_{Ai} - C_{Ai} \\
u_2 &= F - F \\
y' &= C_A - C_A
\end{align*}
\]
Simplify the model with the following parameters:

\[ k = \frac{1}{L} \text{ mol.min} \quad V = 1 L \quad C_{A_{is}} = \frac{2 \text{ mol}}{L} \]

\[ F_s = \frac{1 L}{\text{min}} \quad C_{A_{s}} = \frac{1 \text{ mol}}{L} \]

\[ a_{11} = -1 \frac{L}{L} \cdot \frac{L}{\text{min}} \cdot \frac{1 \text{ mol}}{L} = -1 - 2 = -3 \text{ min}^{-1} \]

\[ b_{11} = \frac{1 L}{\text{min}} = 1 \text{ min}^{-1} \]

\[ b_{12} = \frac{(2 \text{ mol.L} - 1 \text{ mol.L})}{L} = 1 \text{ mol.L}^{-2} \]

\[
\begin{align*}
[\dot{x}_1] &= (-3)[x_1] + \left( \frac{1}{\text{min}^{-1}} \cdot \frac{1 \text{ mol.L}^{-2}}{1} \right) [u_1] \\
[\dot{y}_1] &= [x_1]
\end{align*}
\]
# L.9 Linearization of Model Equations: Examples

2.7 (p. 70)

(a) \[ \frac{dC_{w1}}{dt} = \frac{F}{V_1} (C_{w1} - C_{w1s}) - KC_{w1}^2 \]

\[ \frac{dC_{w2}}{dt} = \frac{F}{V_2} (C_{w1} - C_{w2}) - KC_{w2}^2 \]

(b) @ steady-state, \( C_{w1} = ? \)

\[ C_{w2} = ? \]

\[ \phi = \frac{F}{V_1} (C_{w1s} - C_{w1s}) - k C_{w1s}^2 \]

\[ 1.5 \frac{L}{mol \cdot h} C_{w1s}^2 + 0.25 \frac{C_{w1s}}{h} - 0.25 \left( \frac{1 L}{mol} \right) = \phi \]

\[ C_{w1s} = -0.5, 0.33 \]

\[ C_{w1s} = 0.33 \text{ mol/L} \]

\[ \phi = \frac{0.05}{h} (C_{w2s} - C_{w2s}) - 1.5 C_{w2s}^2 \]

\[ 1.5 C_{w2s}^2 + 0.05 \frac{C_{w2s}}{h} - 0.05 \left( \frac{0.33 \text{ mol}}{L} \right) = \phi \]

\[ C_{w2s} = -0.123, 0.090 \]

\[ C_{w2s} = 0.090 \text{ mol/L} \]

(c) linearize @ steady-state and develop the state space model.

\[ \dot{x} = Ax + Bu \]

\[ x = \begin{bmatrix} C_{w1} - C_{w1s} \\ C_{w2} - C_{w2s} \end{bmatrix} \quad u = \begin{bmatrix} F - F_5 \\ C_{w1} - C_{w1s} \end{bmatrix} \]

\[ a_{ij} = \frac{\partial f_i}{\partial x_j} \bigg|_{s.s.} \quad b_{ij} = \frac{\partial f_i}{\partial u_j} \bigg|_{s.s.} \quad f_1 = \frac{dC_{w1}}{dt} \quad f_2 = \frac{dC_{w2}}{dt} \]

\[ a_{11} = \frac{\partial f_1}{\partial x_1} \bigg|_{s.s.} = \frac{d}{dC_{w1}} \left[ \frac{F}{V_1} (C_{w1} - C_{w1s}) - KC_{w1}^2 \right] = -\frac{F_5}{V_1} - 2kC_{w1s} \]