2nd Order Systems w/o Numerator Dynamics

- Origin: typical example is a 2nd order ODE with constant parameters.

\[ a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_0 u \]

\[ \gamma^2 y'' + 2 \xi \gamma y' + y = ku \quad \text{where:} \quad \gamma^2 = \frac{a_2}{a_0} \]

\[ 2 \xi \gamma = \frac{a_1}{a_0} \]

\[ k = \frac{b_0}{a_0} \]

- \( k = \text{gain} \) [output units/input units]
- \( \xi = \text{damping factor} \) [dimensionless]
- \( \gamma = \text{period} \) [time]

- Take Laplace Transform (assuming: \( y'(0) = y(0) = 0 \))

\[ \gamma^2 s^2 y(s) + 2 \xi \gamma s y(s) + y(s) = k u(s) \]

\[ \Rightarrow y(s) = \frac{k}{\gamma^2 s^2 + 2 \xi \gamma s + 1} \cdot u(s) \]

\[ \Rightarrow \text{2nd order system} \]

- the denominator of our transfer function tells us a lot about our system.
  - roots of denominator = "poles"
  - roots of numerator = "zeros"
2 poles: \[ P_1 = \frac{-\xi}{\zeta} + \frac{\sqrt{\xi^2 - 1}}{\zeta} \]
\[ P_2 = \frac{-\xi}{\zeta} - \frac{\sqrt{\xi^2 - 1}}{\zeta} \]

- It is assumed that \( \xi, \zeta > \phi \), so that the system is stable.
  (by definition \( \zeta > \phi \). Therefore, \( \xi \) must be > \( \phi \) to ensure that the poles are negative).

- There are 3 different cases that we can distinguish, just by looking at the value of \( \xi \) or our "damping factor". (see Table 3-2).

  CASE #1: \( \xi > 1 \)  
  - **POLES**: 2 real, distinct roots  
  - **OUTPUT**: overdamped

  CASE #2: \( \xi = 1 \)  
  - 2 real, equal roots  
  - critically damped

  CASE #3: \( \xi < 1 \)  
  - 2 coupled, complex roots  
  - underdamped

In order to get a better understanding of how these systems behave, we will look at these 3 cases and their response to a step input.

**system model**: \( Y(s) = G_p(s) \cdot U(s) \rightarrow U(s) = \Delta u/s \)
\[ y(s) = \frac{k}{\zeta^2 s^2 + 2\xi \zeta s + 1} \cdot \frac{\Delta u}{s} \]

**CASE #1**: \( \xi > 1 \) (overdamped). Most chemical processes are overdamped.

\[ \text{factor denominator:} \quad \tau_1 = \frac{\zeta}{\xi - \sqrt{\xi^2 - 1}} \quad \tau_2 = \frac{\zeta}{\xi + \sqrt{\xi^2 - 1}} \]


\[ y(s) \rightarrow y(t) : y(t) = k \Delta u \left[ 1 + \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{(\tau_2 - \tau_1)} \right] \]

\[ y(0^+) = ? \quad y(\infty) = ? \]

\[ y(0^+) = \phi \quad y(\infty) = k \Delta u \]

\( \boxed{\text{Higher values of } \phi \text{ tend to approach new steady-state more gradually.}} \)

CASE 2: \( \phi = 1 \) (critically damped)

\( \rightarrow \) Time-domain solution to a step input:

\[ y(t) = k \Delta u \cdot \left[ 1 - (1 + \frac{t}{\tau_2}) e^{-t/\tau_2} \right] \]

\( \boxed{\text{Q: What happened to } \tau_1 \text{ and } \tau_2 ?} \)

\( \boxed{\text{A: } \tau = \tau_1 = \tau_2} \)

\( \boxed{\text{Approach to steady-state is faster than in the overdamped systems.}} \)

\[ \boxed{\text{Comparison of 2nd Order vs. 1st Order system responses:}} \]

- 2nd order systems = have S-shaped responses (with an inflection point at short times.

- 1st order systems = have no inflection point. The maximum slope occurs initially and gradually decreases with time.

\( \boxed{\text{Initial response of the system to a step change is dictated by the "relative order" of the system:}} \)

\[ \text{r.o.} = \text{order of } "s" \text{ in the denominator} - \text{order of } "s" \text{ in numerator} \]

- if r.o. = 1, then initial response has a non-zero slope.

- if r.o. > 1, then initial response has a zero slope.
\[ y(t) = k DU \left[ 1 - \frac{1}{\sqrt{1 - \frac{\xi^2}{r^2}}} \cdot e^{-\xi t / r} \cdot \sin (\alpha t + \phi) \right] \]

\[ \alpha = \sqrt{1 - \frac{\xi^2}{r^2}} \quad \phi = \tan^{-1} \left( \frac{\sqrt{1 - \frac{\xi^2}{r^2}}}{\xi} \right) \]

\[ y(\phi) = k DU \left[ 1 - \frac{1}{\sqrt{1 - \frac{\xi^2}{r^2}}} \cdot \sin (\phi) \right] \]

\[ \sin \phi = \frac{y}{r} \quad \tan \phi = \frac{y}{x} \quad \phi = \tan^{-1} \left( \frac{y}{x} \right) \]

\[ \sin (\tan^{-1} \left( \frac{y}{x} \right)) = \frac{y}{r} \]

\[ y = \sqrt{1 - \frac{\xi^2}{r^2}} \quad x = \frac{\xi}{r} \]

\[ r^2 = x^2 + y^2 = \frac{\xi^2}{r^2} + \sqrt{1 - \frac{\xi^2}{r^2}}^2 = \frac{\xi^2}{r^2} + 1 - \frac{\xi^2}{r^2} = 1 \]

\[ r = \frac{2}{1} \]

\[ \rightarrow y(\phi) = k DU \left[ 1 - \frac{1}{\sqrt{1 - \frac{\xi^2}{r^2}}} \cdot \sqrt{1 - \frac{\xi^2}{r^2}} \right] = k DU \left[ 1 - 1 \right] = 0 \]

**First Order System**: 
\[ y(t) = k DU \left[ 1 - e^{-\xi t / r} \right] \]

**Slope**: 
\[ \frac{dy}{dt} = -k DU \left[ \frac{\xi}{r} \right] e^{-\xi t / r} = \frac{k DU}{r} e^{-\xi t / r} \]

**Slope at t = 0**: 
\[ k DU / r \left( 1 \right) = \frac{k DU}{r} \]
CASE 3: $\xi < 1$ (underdamped):

$$\text{poles} = \frac{-\xi \pm \sqrt{\xi^2 - 1}}{2}$$

Time-domain solution:

$$y(t) = k\Delta u \cdot \left[ 1 - \frac{1}{\sqrt{1 - \xi^2}} \cdot e^{-\xi t/\tau} \cdot \sin \left( \alpha t + \phi \right) \right]$$

$$\alpha = \frac{\sqrt{1 - \xi^2}}{\tau}, \quad \phi = \tan^{-1} \left( \frac{\sqrt{1 - \xi^2}}{\xi} \right)$$

Q: What is the output going to look like?

- $y(\phi) = \text{?}$
- $y(\infty) = \text{?}$

- $y(\phi) = \phi$
- $y(\infty) = k\Delta u$

- As $\xi$ becomes smaller, response is more oscillatory.
- As $\xi$ becomes larger, response is less oscillatory.

- Look at the ratio: Imaginary:\ Real: $\frac{\sqrt{1 - \xi^2}}{\xi}$

$\Rightarrow$ Relative magnitude of oscillations.

**Definitions:**

1. Rise time = amount of time needed to first reach the new steady-state value.
2. Time to first peak = amount of time needed to first reach the first peak.
3. Overshoot = distance between first peak and the new steady-state. Use this to calculate the "overshoot ratio".
4) decay ratio: relative overshoot distance between successive peaks. (measure of how quickly oscillations are decaying).

5) period of oscillation: time between successive peaks.

Now... 2nd Order Systems with Numerator Dynamics

Transfer Function: \( g_p(s) = \frac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)} \) relative order?

This is written in "gain/time constant" form

"pole-zero" form: \( g_p(s) = \frac{K_p (s - z_1)}{(s - p_1)(s - p_2)} \)

Response to a step input:

\[
y(t) = K_p \Delta U \left[ \frac{1}{\tau_1} e^{-t/\tau_1} + \frac{1}{\tau_2} e^{-t/\tau_2} \right]
\]

Example:

\[
y(s) = \frac{\tau_n s + 1}{(3s + 1)(15s + 1)} \cdot \frac{1}{s}
\]

→ Look at limiting cases:

1) \( \tau_n = 3 \) \( \Rightarrow \) \( y(s) = \frac{1}{s(15s + 1)} \)

2) \( \tau_n = 15 \) \( \Rightarrow \) \( y(s) = \frac{1}{s(3s + 1)} \)

☆ These two limiting cases provide a good source of validation, when looking at various values of \( \tau_n \).

☆ \( \tau_n < \phi \) → inverse response. Example: two first-order systems in parallel, with gains of opposite signs.
3.8 Lead-Lag Behavior

Transfer function: \( g_p(s) = \frac{kp}{\frac{\tau_n s + 1}{\tau_p s + 1}} \Rightarrow \frac{s^2 + s + c}{s^2 + s + c} \)

Relative order = ?

Step input response: \( u(s) = \Delta u / s \)

\[ y(t) = kp \Delta u \left[ 1 - \left(1 - \frac{s}{\tau_p} \right) e^{-t/\tau_p} \right] \]

\( y(\phi) = ? \quad y(\infty) = ? \)

\[ y(\phi) = \frac{\tau_n}{\tau_p} \cdot kp \cdot \Delta u \]

\[ y(\infty) = kp \Delta u \]

Chemical Processes do not typically show this type of behavior.

Controllers sometimes do.